

# ON THE SUITA CONJECTURE FOR SOME CONVEX ELLIPSOIDS IN $\mathbb{C}^2$

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ABSTRACT. It has been recently shown that for a convex domain  $\Omega$  in  $\mathbb{C}^n$  and  $w \in \Omega$  the function  $F_\Omega(w) := (K_\Omega(w)\lambda(I_\Omega(w)))^{1/n}$ , where  $K_\Omega$  is the Bergman kernel on the diagonal and  $I_\Omega(w)$  the Kobayashi indicatrix, satisfies  $1 \leq F_\Omega \leq 4$ . While the lower bound is optimal, not much more is known about the upper bound. In general it is quite difficult to compute  $F_\Omega$  even numerically and the highest value of it obtained so far is  $1.010182\dots$ . In this paper we present precise, although rather complicated formulas for the ellipsoids  $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$  (with  $m \geq 1/2$ ) and all  $w$ , as well as for  $\Omega = \{|z_1| + |z_2| < 1\}$  and  $w$  on the diagonal. The Bergman kernel for those ellipsoids had been known, the main point is to compute the volume of the Kobayashi indicatrix. It turns out that in the second case the function  $\lambda(I_\Omega(w))$  is not  $C^{3,1}$ .

## Introduction

For a convex domain  $\Omega$  in  $\mathbb{C}^n$  and  $w \in \Omega$  the following estimates have been recently established:

$$(1) \quad \frac{1}{\lambda(I_\Omega(w))} \leq K_\Omega(w) \leq \frac{4^n}{\lambda(I_\Omega(w))}.$$

Here

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 d\lambda \leq 1\}$$

is the Bergman kernel on the diagonal and

$$I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega), \varphi(0) = w\}$$

is the Kobayashi indicatrix, where  $\Delta$  denotes the unit disc. The first inequality in (1) was shown in [3], the proof uses  $L^2$ -estimates for  $\bar{\partial}$  and Lempert's theory [9]. It is optimal, for example if  $\Omega$  is balanced with respect to  $w$  (that is every intersection of  $\Omega$  with a complex line containing  $w$  is a disc) then we have equality. It can be viewed as a multi-dimensional version of the Suita conjecture [11] proved in [2] (see also [5] for the precise characterization when equality holds).

The second equality in (1) was proved in [4] using rather elementary methods. It was also shown that the constant 4 can be replaced by  $16/\pi^2 = 1.6211\dots$  if  $\Omega$  is in addition symmetric with respect to  $w$ . We can write (1) as

$$1 \leq F_\Omega(w) \leq 4,$$

where  $F_\Omega(w) := (K_\Omega(w)\lambda(I_\Omega(w)))^{1/n}$  is a biholomorphically invariant function in  $\Omega$ . It is not clear what the optimal upper bound should be. It was in fact quite difficult to prove that one can at all have  $F_\Omega > 1$ . It was done in [4] for ellipsoids of the form  $\{|z_1| + |z_2|^{2m} + \dots + |z_n|^{2m} < 1\}$ , where  $m \geq 1/2$  and  $w = (b, 0, \dots, 0)$ . The function  $F_\Omega$  was also computed numerically for the ellipsoid

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$\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ ,  $m \geq 1/2$ , based on an implicit formula for the Kobayashi function from [1]. Our first result is the precise formula in this case:

**Theorem 1.** *For  $m \geq 1/2$  define*

$$\Omega_m = \{z \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}.$$

*Then for  $m \neq 2/3$ ,  $m \neq 2$  and  $b$  with  $0 \leq b < 1$ , we have*

$$\lambda(I_{\Omega_m}((b, 0))) = \pi^2 \left[ -\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} \right. \\ \left. + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m-1}{2m} b^2 + \frac{m}{m+1} \right].$$

*For  $m = 2/3$  and  $m = 2$  one has*

$$\lambda(I_{\Omega_{2/3}}((b, 0))) = \frac{\pi^2}{80} \left( -65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32 \right),$$

$$\lambda(I_{\Omega_2}((b, 0))) = \frac{\pi^2}{240} \left( -3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160 \right).$$

The general formula for the Kobayashi function for  $\Omega_m$  is known, see [1], but it is implicit in the sense that it requires solving a nonlinear equation which is polynomial of degree  $2m$  if it is an integer. It turns out however that the volume of the Kobayashi indicatrix for  $\Omega_m$ , that is the set where the Kobayashi function is not bigger than 1, can be found explicitly. It would be interesting to check whether Theorem 1 also holds in the non-convex case, that is when  $0 < m < 1/2$  (see [10] for computations of the Kobayashi metric in this case).

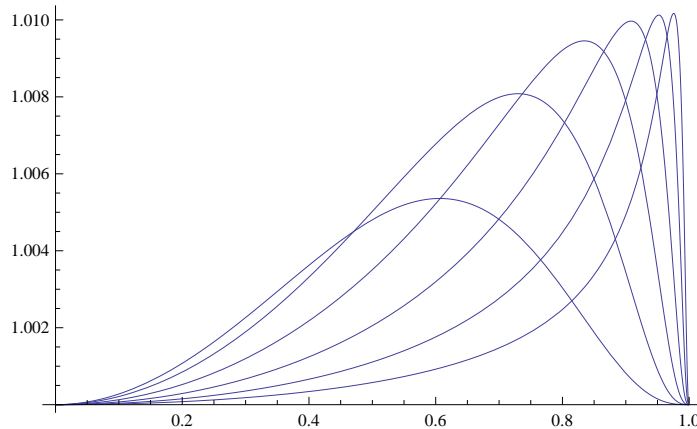
The formula for the Bergman kernel for this ellipsoid is well known (see e.g. [7], Example 6.1.6):

$$K_{\Omega_m}(w) = \frac{1}{\pi^2} (1 - |w_2|^2)^{1/m-2} \frac{(1/m+1)(1 - |w_2|^2)^{1/m} + (1/m-1)|w_1|^2}{((1 - |w_2|^2)^{1/m} - |w_1|^2)^3},$$

so that

$$K_{\Omega_m}((b, 0)) = \frac{m+1 + (1-m)b^2}{\pi^2 m(1-b^2)^3},$$

and we can obtain the following graphs of  $F_{\Omega_m}((b, 0))$  for example for  $m = 4, 8, 16, 32, 64$  and  $128$ :



They are consistent with the graphs from [4] obtained numerically using the implicit formula from [1]. Note that for  $t \in \mathbb{R}$  and  $a \in \Delta$  the mapping

$$\Omega_m \ni z \mapsto \left( e^{it} \frac{(1 - |a|^2)^{1/2m}}{(1 - \bar{a}z_2)^{1/m}} z_1, \frac{z_2 - a}{1 - \bar{a}z_2} \right)$$

is a holomorphic automorphism of  $\Omega_m$  and therefore  $F_{\Omega_m}((b, 0))$  where  $0 \leq b < 1$  attains all values of  $F_{\Omega_m}$  in  $\Omega_m$ . One can show numerically that

$$\sup_{m \geq 1/2} \sup_{\Omega_m} F_{\Omega_m} = 1.010182 \dots$$

which was already noticed in [4]. This is the highest value of  $F_{\Omega}$  (in arbitrary dimension) obtained so far.

In [4] it was also shown that for  $\Omega = \{|z_1| + |z_2| < 1\}$  and  $b$  with  $0 < b < 1$  one has

$$\lambda(I_{\Omega}((b, 0))) = \frac{\pi^2}{6} (1 - b)^4 ((1 - b)^4 + 8b),$$

so that in particular similarly as in Theorem 1 it is an analytic function on this part of  $\Omega$ . This raises a question whether  $\lambda(I_{\Omega}(w))$  is smooth in general. In [4] it was also predicted that the highest value of  $F_{\Omega}$  for convex  $\Omega$  in  $\mathbb{C}^2$  should be attained for  $\Omega = \{|z_1| + |z_2| < 1\}$  on the diagonal. The following result will answer both of these questions in the negative:

**Theorem 2.** *Let  $\Omega = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ . Then for  $b$  with  $0 \leq b \leq 1/4$  we have*

$$(2) \quad \lambda(I_{\Omega}((b, b))) = \frac{\pi^2}{6} (30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1)$$

and when  $1/4 \leq b < 1/2$

$$(3) \quad \begin{aligned} \lambda(I_{\Omega}((b, b))) &= \frac{2\pi^2 b(1 - 2b)^3 (-2b^3 + 3b^2 - 6b + 4)}{3(1 - b)^2} \\ &+ \frac{\pi (30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1)}{6(1 - b)^2} \\ &\quad \times \arccos \left( -1 + \frac{4b - 1}{2b^2} \right) \\ &+ \frac{\pi(1 - 2b) (-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37)}{72(1 - b)} \sqrt{4b - 1} \\ &+ \frac{4\pi b(1 - 2b)^4 (7b^2 + 2b - 2)}{3(1 - b)^2} \arctan \sqrt{4b - 1} \\ &+ \frac{4\pi b^2(1 - 2b)^4(2 - b)}{(1 - b)^2} \arctan \frac{1 - 3b}{(1 - b)\sqrt{4b - 1}}. \end{aligned}$$

The function

$$b \mapsto \lambda(I_{\Omega}((b, b)))$$

is  $C^3$  on the interval  $(0, 1/2)$  but not  $C^{3,1}$  at  $1/4$ .

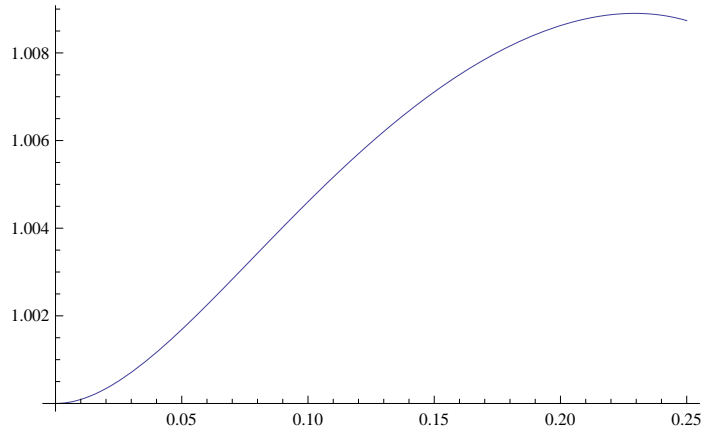
Again, the formula for the Bergman metric for this ellipsoid is known, see [6] or [7], Example 6.1.9:

$$K_{\Omega}(w) = \frac{2}{\pi^2} \cdot \frac{3(1 - |w|^2)^2(1 + |w|^2) + 4|w_1|^2|w_2|^2(5 - 3|w|^2)}{((1 - |w|^2)^2 - 4|w_1|^2|w_2|^2)^3},$$

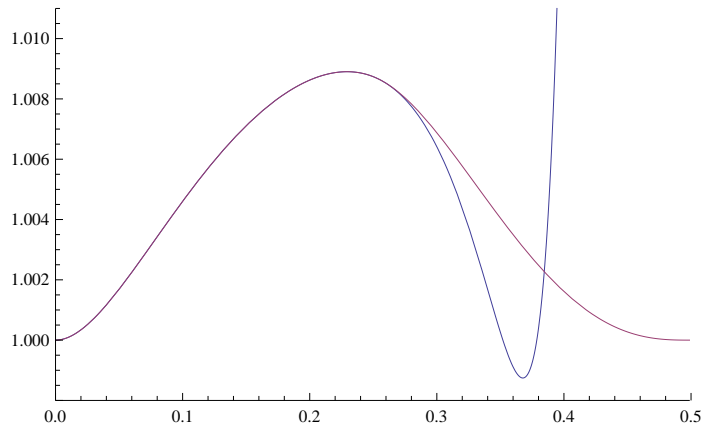
so that

$$(4) \quad K_{\Omega}((b, b)) = \frac{2(3 - 6b^2 + 8b^4)}{\pi^2(1 - 4b^2)^3}.$$

The first part of Theorem 2, formula (2) on the interval  $(0, 1/4)$ , is easier to prove than the second one. Combining it with (4) one obtains the following graph of  $F_{\Omega}((b, b))$  for  $b \in (0, 1/4)$ :



One can show that its analytic continuation to  $(0, 1/2)$  attains values below 1 and thus it follows already from (1) that  $F_{\Omega}$  cannot be analytic. To conclude that it is in fact not  $C^{3,1}$  one has to prove much harder formula (3). Here is the full picture on the interval  $(0, 1/2)$ , the analytic continuation of  $F_{\Omega}$  from  $(0, 1/4)$  and the actual graph of  $F_{\Omega}$ :



One can check that the maximal value of  $F_{\Omega}((b, b))$  for  $b \in (0, 1/2)$  is  $1.008902\dots$

All pictures and numerical computations in this paper, as well as a lot of formal ones in the proofs of Theorems 1 and 2 have been done using *Mathematica*.

### 1. General formula for geodesics in convex complex ellipsoids

Boundary of the Kobayashi indicatrix of a convex domain  $\Omega$  at  $w$  consists of the vectors  $\varphi'(0)$  where  $\varphi \in \mathcal{O}(\Delta, \Omega)$  is a geodesic of  $\Omega$  satisfying  $\varphi(0) = w$ . Theorems 1 and 2 will be proved using a general formula for geodesics in convex complex ellipsoids from [8] based on Lempert's theory [9] describing geodesics of smooth strongly convex domains.

For  $p = (p_1, \dots, p_n)$  with  $p_j \geq 1/2$  set

$$\mathcal{E}(p) = \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \dots + |z_n|^{2p_n} < 1\}$$

and  $A \subset \{1, \dots, n\}$  define

$$\varphi_j(\zeta) = \begin{cases} a_j \frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta} \left( \frac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/p_j}, & j \in A \\ a_j \left( \frac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/p_j}, & j \notin A \end{cases},$$

where  $a_j \in \mathbb{C}_*$ ,  $\alpha_0, \alpha_j \in \Delta$  for  $j \in A$ ,  $\alpha_j \in \bar{\Delta}$  for  $j \notin A$ ,

$$(5) \quad \alpha_0 = |a_1|^{2p_1} \alpha_1 + \dots + |a_n|^{2p_n} \alpha_n,$$

and

$$(6) \quad 1 + |\alpha_0|^2 = |a_1|^{2p_1} (1 + |\alpha_1|^2) + \dots + |a_n|^{2p_n} (1 + |\alpha_n|^2).$$

A component  $\varphi_j$  has a zero in  $\Delta$  if and only if  $j \in A$ . We have

$$(7) \quad \varphi_j(0) = \begin{cases} -a_j \alpha_j, & j \in A \\ a_j, & j \notin A \end{cases},$$

and

$$(8) \quad \varphi'_j(0) = \begin{cases} a_j \left( 1 + \left( \frac{1}{p_j} - 1 \right) |\alpha_j|^2 - \frac{\alpha_j \bar{\alpha}_0}{p_j} \right), & j \in A \\ a_j \frac{\bar{\alpha}_0 - \bar{\alpha}_j}{p_j}, & j \notin A \end{cases}.$$

For  $w \in \mathcal{E}(p)$  the set of vectors  $\varphi'(0)$  where  $\varphi(0) = w$  forms a subset of  $\partial I_{\mathcal{E}(p)}^K(w)$  of a full measure. The geodesics in  $\mathcal{E}(p)$  are uniquely determined: for a given  $w \in \mathcal{E}(p)$  and  $X \in (\mathbb{C}^n)_*$  there exists unique geodesic  $\varphi \in \mathcal{O}(\Delta, \mathcal{E}(p))$  such that  $\varphi(0) = w$  and  $\varphi'(0) = X$ .

### 2. Proof of Theorem 1

First note that the formulas for  $m = 2/3$  and  $m = 2$  easily follow from the first one by approximation. For  $\Omega_m = \mathcal{E}(m, 1)$  and  $w = (b, 0)$  there are two possibilities for a geodesic  $\varphi$ : either  $\varphi$  crosses the axis  $\{z_1 = 0\}$  or it does not. By  $I_{12}$  and  $I_2$  denote the respective parts of  $I_{\Omega_m}(w)$ . In the first case  $\varphi$  must be of the form

$$\varphi(\zeta) = \left( a_1 \frac{\zeta - \alpha_1}{1 - \bar{\alpha}_1 \zeta} \left( \frac{1 - \bar{\alpha}_1 \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/m}, a_2 \frac{\zeta - \alpha_2}{1 - \bar{\alpha}_0 \zeta} \right),$$

where  $a_1, a_2 \in \mathbb{C}_*$  and  $\alpha_0, \alpha_1, \alpha_2 \in \Delta$  satisfy (5), (6). By (7) and since  $\varphi(0) = (b, 0)$  we have  $a_1 = -b/\alpha_1$ ,  $\alpha_2 = 0$  and by (5)  $\alpha_0 = b^{2m}\alpha_1/|\alpha_1|^{2m}$ . By (6)

$$1 + b^{4m}|\alpha_1|^{2-4m} = b^{2m}|\alpha_1|^{-2m} (1 + |\alpha_1|^2) + |a_2|^2,$$

that is

$$(9) \quad |a_2|^2 = (1 - b^{2m}|\alpha_1|^{-2m})(1 - b^{2m}|\alpha_1|^{2-2m}).$$

Since  $\alpha_0, \alpha_1 \in \Delta_*$ , it follows that  $b < |\alpha_1| < 1$ . Write  $\alpha_1 = -re^{-it}$ ,  $a_2 = \rho e^{is}$ , then by (8) and (9), with  $b < r < 1$ ,

$$\begin{aligned} \varphi'(0) &= \left( \left( \frac{b}{r} + b \left( \frac{1}{m} - 1 \right) r - \frac{b^{2m+1}r^{1-2m}}{m} \right) e^{it}, \sqrt{(1 - b^{2m}r^{-2m})(1 - b^{2m}r^{2-2m})} e^{is} \right) \\ &=: (\gamma_1(r)e^{it}, \gamma_2(r)e^{is}). \end{aligned}$$

The mapping

$$(10) \quad \Delta \times [0, 2\pi) \times (b, 1) \ni (\zeta, t, r) \mapsto \zeta(\gamma_1(r)e^{it}, \gamma_2(r))$$

parametrizes  $I_{12}$ . We will need a lemma.

**Lemma 3.** *Let  $F(\zeta, z) = \zeta(f(z), g(z))$  be a function of two complex variables, where  $f$  and  $g$  are  $C^1$ . Then the real Jacobian of  $F$  is equal to  $|\zeta|^2 H(z)$ , where*

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\operatorname{Re}(f\bar{g}(\bar{f}_z g_z - \bar{f}_{\bar{z}} g_{\bar{z}})).$$

The proof is left to the reader. For the mapping (10) we can compute that

$$\begin{aligned} H &= \gamma_1 \gamma_2 (\gamma_1' \gamma_2' - \gamma_1' \gamma_2') \\ &= -\frac{b^2}{m^2} r^{-6m-3} [b^{2m}(-mr^2 + m - 1) + r^{2m}] [r^{2m}((m-1)r^2 + m) - (2m-1)r^2 b^{2m}] \\ &\quad \times [r^2 b^{2m} + r^{2m}((m-1)r^2 - m)] \end{aligned}$$

Since

$$(11) \quad \int_{\Delta} |\zeta|^2 d\lambda(\zeta) = \frac{\pi}{2},$$

we obtain

$$\begin{aligned} \lambda(I_{12}) &= \pi^2 \int_b^1 |H| dr \\ (12) \quad &= \pi^2 \left( \frac{(1-2m)^2}{m^2(3m-1)(3m-2)} b^{6m+2} - \frac{3}{m^2(m+1)(m-2)} b^{2m+2} - \frac{3}{2m^2} b^{4m+2} \right. \\ &\quad \left. + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m^2-m+1}{2m^2} b^2 + \frac{m}{m+1} \right). \end{aligned}$$

To compute the volume of  $I_2$  we consider geodesics of the form

$$\varphi(\zeta) = \left( a_1 \left( \frac{1 - \bar{\alpha}_1 \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/m}, a_2 \frac{\zeta - \alpha_2}{1 - \bar{\alpha}_0 \zeta} \right),$$

where  $a_1, a_2 \in \mathbb{C}_*$ ,  $\alpha_0, \alpha_2 \in \Delta$ ,  $\alpha_1 \in \bar{\Delta}$  satisfy (5), (6). By (7) and since  $\varphi(0) = (b, 0)$  we have  $a_1 = b$ ,  $\alpha_2 = 0$  and by (5)  $\alpha_0 = b^{2m}\alpha_1$ . By (6)

$$1 + b^{4m}|\alpha_1|^2 = b^{2m} (1 + |\alpha_1|^2) + |a_2|^2,$$

that is

$$|a_2|^2 = (1 - b^{2m})(1 - b^{2m}|\alpha_1|^2).$$

This means that any  $\alpha_1 \in \Delta$  is allowed and by (8)

$$\begin{aligned} \varphi'(0) &= \left( \frac{b(b^{2m} - 1)}{m} \bar{\alpha}_1, a_2 \right) \\ &= \left( \frac{b(1 - b^{2m})r}{m} e^{it}, \sqrt{(1 - b^{2m})(1 - b^{2m}r^2)} e^{is} \right), \end{aligned}$$

where  $\alpha_1 = -re^{-it}$ ,  $a_2 = \rho e^{is}$ . Similarly as before we have

$$H = -\frac{b^2(1 - b^{2m})^3 r}{m^2}$$

and

$$\lambda(I_2) = \pi^2 \int_0^1 |H| dr = \frac{\pi^2 b^2 (1 - b^{2m})^3}{2m^2}.$$

This combined with (12) finishes the proof of Theorem 1.  $\square$

### 3. Proof of Theorem 2

For  $\Omega = \mathcal{E}(1/2, 1/2)$  and  $w = (b, b)$ , where  $0 < b < 1/2$ , we have by (7)

$$(13) \quad a_j = \begin{cases} -\frac{b}{\alpha_j}, & j \in A \\ b, & j \notin A \end{cases}$$

and by (8)

$$(14) \quad \varphi'_j(0) = \begin{cases} 2b\bar{\alpha}_0 - b \left( \bar{\alpha}_j + \frac{1}{\alpha_j} \right), & j \in A \\ 2b(\bar{\alpha}_0 - \bar{\alpha}_j), & j \notin A \end{cases}.$$

There are four possibilities for the set  $A$ :  $\emptyset$ ,  $\{1\}$ ,  $\{2\}$ , and  $\{1, 2\}$ . Denote the corresponding parts of  $I_\Omega(w)$  by  $I_0$ ,  $I_1$ ,  $I_2$ , and  $I_{12}$ , respectively, so that

$$(15) \quad \begin{aligned} \lambda(I_\Omega(w)) &= \lambda(I_0) + \lambda(I_1) + \lambda(I_2) + \lambda(I_{12}) \\ &= \lambda(I_0) + 2\lambda(I_1) + \lambda(I_{12}). \end{aligned}$$

**The case  $A = \{1, 2\}$**

By (5), (6) and (13)

$$(16) \quad \left( \frac{1}{b} + 2b \right) |\alpha_1| |\alpha_2| + 2b \operatorname{Re}(\alpha_1 \bar{\alpha}_2) = (1 + |\alpha_1|^2) |\alpha_2| + (1 + |\alpha_2|^2) |\alpha_1|.$$

Since the set of  $\alpha \in \Delta^2$  satisfying (16) is  $S^1$ -invariant, let us consider only those  $\alpha$  with  $\alpha_2 > 0$ . If we then replace  $\alpha_1$  with  $\bar{\alpha}_1$  then (16) will still be valid and  $\varphi'(0)$  will be replaced by  $\overline{\varphi'(0)}$ . We thus consider

$$(17) \quad \alpha_1 = re^{it}, \quad \alpha_2 = \rho, \quad r, \rho \in (0, 1), \quad t \in (0, \pi);$$

to get  $\lambda(I_{12})$  we will have to multiply the obtained volume by 2. The condition (16) transforms to

$$(18) \quad \frac{1}{b} + 2b(1 + \cos t) = r + \frac{1}{r} + \rho + \frac{1}{\rho}.$$

It will be convenient to substitute  $x = r + 1/r$ ,  $y = t$ , and consider the domain

$$(19) \quad U := \left\{ (x, y) \in \left(2, \frac{1}{b} + 4b - 2\right) \times (0, \pi) : x < \frac{1}{b} + 2b(1 + \cos y) - 2 \right\}.$$

We have

$$\alpha_0 = b \left( \frac{\alpha_1}{|\alpha_1|} + \frac{\alpha_2}{|\alpha_2|} \right) = b(e^{it} + 1)$$

and thus by (14) and (18)

$$(20) \quad \begin{aligned} \varphi'(0) &= b \left( 2\bar{\alpha}_0 - \bar{\alpha}_1 - \frac{1}{\alpha_1}, 2\bar{\alpha}_0 - \bar{\alpha}_2 - \frac{1}{\alpha_2} \right) \\ &= \left( 2b^2(e^{-it} + 1) - b\left(r + \frac{1}{r}\right)e^{-it}, 2b^2(e^{-it} + 1) - b\left(\rho + \frac{1}{\rho}\right) \right) \\ &= (2b^2 + b(2b - x)e^{-iy}, bx - 1 - 2b^2i \sin y) \\ &=: (f(z), g(z)). \end{aligned}$$

The mapping

$$\Delta \times U \ni (\zeta, z) \mapsto \zeta(f(z), g(z))$$

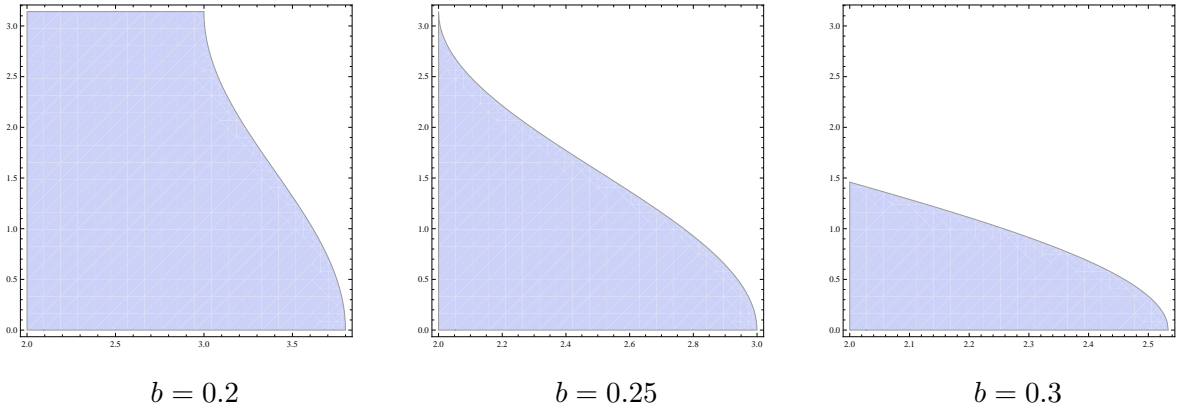
parametrizes  $I_{12}$ . From Lemma 3 and (11) it follows that

$$\lambda(I_{12}) = \pi \iint_U |H| d\lambda,$$

where  $f, g$  are given by (20),  $U$  by (19) (recall that again we had to multiply by 2) and we can compute that

$$H = b^2 [1 - 2b^2(\cos y + 1)] [-bx^2 + (1 + 2b^2(\cos y + 1))(x - 2b) - 2b(b^2 \cos(2y) + 1)].$$

One can check that  $H > 0$  in  $U$ . The region  $U$  may look as follows



We set

$$y_0 := \begin{cases} \pi & b \leq 1/4 \\ \arccos \left( -1 + \frac{4b-1}{2b^2} \right) & b > 1/4 \end{cases},$$



then

$$\lambda(I_{12}) = \pi \int_0^{y_0} \int_2^{1/b+2b(1+\cos y)-2} H \, dx dy.$$

For  $b \leq 1/4$  we will get

$$(21) \quad \lambda(I_{12}) = \frac{\pi^2}{6}(1 - 32b^2 + 80b^3 - 12b^4 - 112b^5 + 176b^6 - 192b^7 + 110b^8)$$

and for  $b > 1/4$

$$(22) \quad \begin{aligned} \lambda(I_{12}) &= \frac{\pi}{72}(37 - 140b + 270b^2 - 528b^3 + 530b^4 - 712b^5 + 660b^6)(1 - 2b)\sqrt{4b - 1} \\ &+ \frac{\pi}{6}(1 - 32b^2 + 80b^3 - 12b^4 - 112b^5 + 176b^6 - 192b^7 + 110b^8) \arccos\left(-1 + \frac{4b - 1}{2b^2}\right). \end{aligned}$$

**The case  $A = \{1\}$**

By (13)  $a_1 = -b/\alpha_1$ ,  $a_2 = b$  and by (5)  $\alpha_0 = b(\alpha_1/|\alpha_1| + \alpha_2)$ . From (6) we get

$$(23) \quad 1 + b^2 \left(1 + \frac{2\operatorname{Re}(\alpha_1 \bar{\alpha}_2)}{|\alpha_1|} + |\alpha_2|^2\right) = \frac{b}{|\alpha_1|}(1 + |\alpha_1|^2) + b(1 + |\alpha_2|^2).$$

We may assume that  $\alpha_1 > 0$ , then (23) has a solution  $\alpha_1 \in (0, 1)$  if and only if  $T > 2$ , where

$$\begin{aligned} T &= \frac{1}{b} + b(1 + 2\operatorname{Re} \alpha_2 + |\alpha_2|^2) - 1 - |\alpha_2|^2 \\ &= \frac{1}{b} + b - 1 + 2bx - (1 - b)(x^2 + y^2), \end{aligned}$$

and we write  $\alpha_2 = x + iy$ . This means that

$$(24) \quad \left| \alpha_2 - \frac{b}{1 - b} \right| < \frac{1 - 2b}{\sqrt{b}(1 - b)}$$

and the set  $U$  will be the intersection of this disc with  $\Delta$ . By (14) and (23)

$$\varphi'(0) = 2b(b(1 + \bar{\alpha}_2) - T/2, b - (1 - b)\bar{\alpha}_2)$$

and therefore

$$\begin{aligned} f &= 2b^2(1 + x) - bT - 2b^2yi, \\ g &= 2b^2 - 2b(1 - b)x + 2b(1 - b)yi. \end{aligned}$$

We can compute that

$$\begin{aligned} H &= 4(1 - b)b^2[b^2(1 + 2x) - (1 - b)(1 + b(x^2 + y^2))] \\ &\quad [-1 + 2b + b^3 - 2b^2(1 - b)x + b(1 - b)^2(x^2 + y^2)] \\ &= 4(1 - b)b^3(b + b^2 - (1 - b)T)(b^2 + 2b - 2 + bT). \end{aligned}$$

One can check that  $H > 0$  everywhere on  $U$ .

If  $b \leq 1/4$  then  $U = \Delta$  and using the polar coordinates in  $\Delta$  and Lemma 3 we will get

$$(25) \quad \lambda(I_1) = \frac{2\pi^2}{3}(1 - b)b^2(3 - 9b + 2b^2 + 6b^3 - 6b^4 + 10b^5).$$

For  $b > 1/4$  it is more convenient to use the polar coordinates in the disk (24) instead:

$$x = \frac{b}{1 - b} + r \cos t, \quad y = r \sin t,$$

then

$$H = 4b^2(1 - 2b)^2 - 4b^4(1 - b)^4 r^4.$$

For  $r$  with

$$\frac{1 - 2b}{1 - b} < r < \frac{1 - 2b}{\sqrt{b}(1 - b)}$$

the circles  $\{|\alpha_2 - b/(1 - b)| = r\}$  and  $\{|\alpha_2| = 1\}$  intersect when  $t = \pm t(r)$ , where

$$(26) \quad t(r) = \arccos \frac{1 - 2b - (1 - b)^2 r^2}{2br(1 - b)}.$$

Therefore

$$\lambda(I_1) = \pi^2 \int_0^{(1-2b)/(1-b)} r H dr + \pi \int_{(1-2b)/(1-b)}^{(1-2b)/(\sqrt{b}(1-b))} r(\pi - t(r)) H dr.$$

We can compute the second integral using the following indefinite integrals:

$$(27) \quad \begin{aligned} \int v \arccos\left(\frac{a}{v} - v\right) dv &= \frac{1}{4} \sqrt{-a^2 + 2av^2 - v^4 + v^2} \\ &+ \frac{4a + 1}{8} \arctan \frac{2a - 2v^2 + 1}{2\sqrt{-a^2 + 2av^2 - v^4 + v^2}} + \frac{v^2}{2} \arccos\left(\frac{a}{v} - v\right) + \text{const}, \\ \int v^5 \arccos\left(\frac{a}{v} - v\right) dv &= \frac{1}{288} (15 + 78a + 80a^2 + (10 + 32a)v^2 + 8v^4) \sqrt{-a^2 + 2av^2 - v^4 + v^2} \\ &+ \frac{5 + 36a + 72a^2 + 32a^3}{192} \arctan \frac{2a - 2v^2 + 1}{2\sqrt{-a^2 + 2av^2 - v^4 + v^2}} + \frac{v^6}{6} \arccos\left(\frac{a}{v} - v\right) + \text{const}. \end{aligned}$$

We will obtain

$$(28) \quad \begin{aligned} \lambda(I_1) &= - \frac{\pi^2 b (10b^9 - 36b^8 + 54b^7 + 84b^6 - 375b^5 + 414b^4 - 166b^3 - 6b^2 + 21b - 4)}{3(1 - b)^2} \\ &+ \frac{\pi b(1 - 2b) (30b^6 - 58b^5 + 43b^4 - 19b^3 - 26b^2 + 32b - 8)}{9(1 - b)} \sqrt{4b - 1} \\ &+ \frac{4\pi(1 - 2b)^4 b (2b^2 - 2b - 1)}{3(1 - b)^2} \arccos \frac{3b - 1}{2b^{3/2}} \\ &+ \frac{2}{3} \pi(1 - b)b^2 (10b^5 - 6b^4 + 6b^3 + 2b^2 - 9b + 3) \arctan \frac{2b^2 - 4b + 1}{(1 - 2b)\sqrt{4b - 1}}. \end{aligned}$$

for  $b > 1/4$ .

**The case  $A = \emptyset$**

We have  $a_1 = a_2 = b$  and  $\alpha_0 = b(\alpha_1 + \alpha_2)$ . Therefore

$$(29) \quad -b(1 - b)(|\alpha_1|^2 + |\alpha_2|^2) + 2b^2 \operatorname{Re}(\alpha_1 \bar{\alpha}_2) + 1 - 2b = 0.$$

Again, we may assume that  $\alpha_1 > 0$ . We may also assume that  $\operatorname{Re} \alpha_2 \geq 0$  and then multiply the resulting integral by 2. The equation (29) has a solution  $\alpha_1$  if

$$D := -b(1 - b)^2 |\alpha_2|^2 + b^3 (\operatorname{Re} \alpha_2)^2 + (1 - b)(1 - 2b) \geq 0.$$

It satisfies  $\alpha_1 < 1$  if

$$Q := \frac{b^{3/2}\operatorname{Re} \alpha_2 + \sqrt{D}}{\sqrt{b}(1-b)} < 1.$$

This means that

$$(30) \quad \left| \alpha_2 - \frac{b}{1-b} \right| > \frac{1-2b}{\sqrt{b}(1-b)}.$$

By  $U$  we will denote the set of  $\alpha_2 \in \Delta$  satisfying (30). For  $b \leq 1/4$  we have  $U = \emptyset$  and thus  $\lambda(I_0) = 0$  then. This together with (15), (21) and (25) gives (2).

Assume that  $b > 1/4$ . By (14)

$$\varphi'(0) = 2b((b-1)Q + b\bar{\alpha}_2, bQ + (b-1)\bar{\alpha}_2),$$

so that,

$$\begin{aligned} f &= 2b((b-1)Q + bx) - 2b^2y i \\ g &= 2b(bQ + (b-1)x) + 2b(1-b)y i. \end{aligned}$$

One can compute that

$$H = \frac{16b^3(1-2b)^3}{1-b} \left( 1 + \frac{b^{3/2}x}{\sqrt{D}} \right).$$

By Lemma 3

$$\lambda(I_0) = \pi \int_{-1}^{-1+(4b-1)/(2b^2)} \int_{y_2(x)}^{\sqrt{1-x^2}} H dy dx,$$

where

$$y_2(x) = \begin{cases} 0, & -1 \leq x \leq \frac{b^{3/2} + 2b - 1}{\sqrt{b}(1-b)} \\ \sqrt{\frac{(1-2b)^2}{b(1-b)^2} - \left(x - \frac{b}{1-b}\right)^2}, & \frac{b^{3/2} + 2b - 1}{\sqrt{b}(1-b)} \leq x \leq -1 + \frac{4b-1}{2b^2}. \end{cases}$$

It is clear from this formula that  $\lambda(I_0)$  is analytic for  $b \in (1/4, 1/2)$ . We may therefore restrict ourselves to the interval  $(1/4, (3-\sqrt{5})/2)$ , then  $0 \notin U$  and we will use polar coordinates in  $\Delta$ , that is

$$x = r \cos t, \quad y = r \sin t.$$

We will get

$$\begin{aligned} \lambda(I_0) &= \frac{16\pi b^3(1-2b)^3}{1-b} \int_{r_0}^1 r \left( \arccos \frac{1-3b+b^2-b(1-b)r^2}{2b^2r} \right. \\ &\quad \left. - \arctan \frac{\sqrt{4b^4r^2 - (1-3b+b^2-b(1-b)r^2)^2}}{1-b-b^2-b(1-b)r^2} \right) dr, \end{aligned}$$

where

$$r_0 = \frac{1-2b-b^{3/2}}{\sqrt{b}(1-b)}.$$

Using (27) one can compute that

$$\int_{r_0}^1 r \arccos \frac{1-3b+b^2-b(1-b)r^2}{2b^2r} dr = \frac{\pi(2b^3-8b^2+6b-1)}{4(b-1)^2b} - \frac{1}{2} \arccos \left( -1 + \frac{4b-1}{2b^2} \right) + \frac{1-2b}{4b(1-b)} \sqrt{4b-1} + \frac{(1-2b)^2}{2b(1-b)^2} \arctan \frac{1-3b}{(1-b)\sqrt{4b-1}}.$$

On the other hand, since

$$\int \frac{1}{v^2} \arctan \sqrt{-av^2+v-1} dv = \frac{1}{2v} \sqrt{-av^2+v-1} - \frac{1}{v} \arctan \sqrt{-av^2+v-1} - \frac{a}{2} \arctan \frac{2a\sqrt{-av^2+v-1}}{-av-2a+1} + \frac{2a-1}{4} \arctan \frac{(v-2)\sqrt{-av^2+v-1}}{2av^2-2v+2} + \text{const},$$

we will obtain

$$\int_{r_0}^1 r \arctan \frac{\sqrt{4b^4r^2-(1-3b+b^2-b(1-b)r^2)^2}}{1-b-b^2-b(1-b)r^2} dx = \frac{\pi(1-2b)(b+1)}{8(1-b)^2} + \frac{1-2b}{4b(1-b)} \sqrt{4b-1} - \frac{(b+2)(1-2b)}{4b(1-b)} \arctan \sqrt{4b-1} - \frac{(1+b)(1-2b)}{4(1-b)^2} \arctan \frac{1-3b}{(1-b)\sqrt{4b-1}}.$$

Therefore

$$(31) \quad \lambda(I_0) = \frac{2\pi^2b^2(1-2b)^3(-6b^2+9b-2)}{(1-b)^2} - \frac{8\pi b^3(1-2b)^3}{1-b} \arccos \left( -1 + \frac{4b-1}{2b^2} \right) + \frac{4\pi b^2(1-2b)^4(b+2)}{(1-b)^2} \arctan \sqrt{4b-1} + \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2} \arctan \frac{1-3b}{(1-b)\sqrt{4b-1}}.$$

Using the formulas

$$(32) \quad \arccos \left( -1 + \frac{4b-1}{2b^2} \right) = \arctan \frac{2b^2-4b+1}{(1-2b)\sqrt{4b-1}} + \frac{\pi}{2},$$

$$\arccos \frac{3b-1}{2b^{3/2}} = \arctan \sqrt{4b-1} - \arctan \frac{2b^2-4b+1}{(1-2b)\sqrt{4b-1}} + \frac{\pi}{2},$$

and combining (15), (22), (28) and (31), we get (3) for  $b > 1/4$ .

Denoting by  $\chi_-$  and  $\chi_+$  the functions defined by the right-hand sides of (2) and (3), respectively, we can compute that at  $1/4$

$$\chi_- = \chi_+ = \frac{15887}{196608} \pi^2, \quad \chi'_- = \chi'_+ = -\frac{3521}{6144} \pi^2, \quad \chi''_- = \chi''_+ = -\frac{215}{1536} \pi^2, \quad \chi_-^{(3)} = \chi_+^{(3)} = \frac{1785}{64} \pi^2,$$

but

$$\chi_-^{(4)} = \frac{1549}{16} \pi^2, \quad \chi_+^{(4)} = \infty.$$

This shows that our function is  $C^3$  but not  $C^{3,1}$  at  $1/4$ . □

In fact, using (32) and

$$\arctan(1/x) = \frac{\pi}{2} - \arctan x, \quad x > 0,$$

for  $b \in (1/4, 1 - 1/\sqrt{2})$  the formula (3) can be written as

$$\begin{aligned} \lambda(I_\Omega((b, b))) &= \frac{\pi^2}{6} (30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1) \\ &+ \frac{\pi(1-2b)(-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37)}{72(1-b)} \sqrt{4b-1} \\ &+ \frac{4\pi b(1-2b)^4(7b^2+2b-2)}{3(1-b)^2} \arctan \sqrt{4b-1} \\ &+ \frac{\pi(30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1)}{6(1-b)^2} \\ &\quad \times \arctan \frac{(1-2b)\sqrt{4b-1}}{2b^2 - 4b + 1} \\ &- \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2} \arctan \frac{(1-b)\sqrt{4b-1}}{1-3b}. \end{aligned}$$

#### REFERENCES

- [1] B.E. BLANK, D. FAN, D. KLEIN, S.G. KRANTZ, D. MA, M.-Y. PANG, *The Kobayashi metric of a complex ellipsoid in  $\mathbb{C}^2$* , Experimental Math. 1 (1992), 47–55
- [2] Z. BŁOCKI, *Suita conjecture and the Ohsawa-Takegoshi extension theorem*, Invent. Math. 193 (2013), 149–158
- [3] Z. BŁOCKI, *A lower bound for the Bergman kernel and the Bourgain-Milman inequality*, GAFA Seminar Notes, Lect. Notes in Math., Springer (to appear)
- [4] Z. BŁOCKI, W. ZWONEK, *Estimates for the Bergman kernel and the multidimensional Suita conjecture*, arXiv:1404.7692
- [5] Q. GUAN, X. ZHOU, *A solution of an  $L^2$  extension problem with optimal estimate and applications*, arXiv:1310.7169v4, Ann. of Math. (to appear)
- [6] K.T. HAHN, P. PFLUG, *The Kobayashi and Bergman metrics on generalized Thullen domains*, Proc. Amer. Math. Soc. 104 (1988), 207–214
- [7] M. JARNICKI, P. PFLUG, *Invariant Distances and Metrics in Complex Analysis*, 2nd ext. ed., Walter de Gruyter, 2013
- [8] M. JARNICKI, P. PFLUG, R. ZEINSTRAS, *Geodesics for convex complex ellipsoids*, Ann. Scuola Norm. Sup. Pisa 20 (1993), 535–543
- [9] L. LEMPET, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474
- [10] P. PFLUG, W. ZWONEK, *The Kobayashi metric for non-convex complex ellipsoids*, Complex Variables Theory Appl. 29 (1996), 59–71
- [11] N. SUITA, *Capacities and kernels on Riemann surfaces*, Arch. Ration. Mech. Anal. 46 (1972), 212–217

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